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On Some Classes of Generalized Random Linear Functionals[1]

Z. LOZANOV-CRVENKOVIĆ AND S. PILIPOVIĆ

Institute of Mathematics, University of Novi Sad, Ilije Djuričića 4, 21000 Novi Sad, Yugoslavia

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Representation theorems for some new classes of random linear functionals and an application of these results in solving stochastic evolution equations are given.

1. INTRODUCTION

Let $(\Omega, \mathfrak{A}, \mu)$ be a probability space and V be a topological vector space. A complex valued function f on $\Omega \times V$, $f: (\omega, v) \mapsto f(\omega, v)$ is called a random functional if $f(\cdot, v)$ is a random variable for all $v \in V$. In the papers [7, 6, 1], the spaces $\mathscr{D}, \mathscr{K}\{M_p\}$, $\mathscr{S}\{M_p\}$ were taken to be V, and the random functional was considered as a mapping from Ω to $\mathscr{D}'$, $\mathscr{K}'\{M_p\}$, $\mathscr{S}'\{M_p\}$, respectively. Thus these random functionals are for any $\omega \in \Omega$ continuous linear functionals on the spaces $\mathscr{D}$, $\mathscr{K}\{M_p\}$, and $\mathscr{S}\{M_p\}$, respectively. Using the methods of functional analysis and the probability theory, in mentioned papers, representation theorems for corresponding random linear functions (rlf) were obtained. In [6] some examples of rlf were given as well.

In this paper we shall give representation theorems for some new classes of random linear functionals; for V we shall take the testing functions spaces $\mathscr{D}^{(M_p)}(\mathcal{O})$ [2], and $\operatorname{Exp} \mathscr{A}$ [4]. An application of these results in solving stochastic evolution equations is given.

2. FUNCTION SPACES

We shall recall some basic definitions from [2]. Let $\mathcal{O}$ be an open subset of $\mathfrak{R}^n$ and K be a regular compact set in $\mathcal{O}$ in the sense of Whitney. This

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means that K is a compact set in $\mathfrak{R}^n$ with the following property: there exists $c>0$ such that any two points x and y of any connected component $L \subset K$ are joined by an arc in L of length less than or equal to $c|x-y|$. Throughout the paper K will always denote a regular compact set. Let M_p, $p \in \mathfrak{N} = \{1, 2, ...\}$ be a sequence of positive numbers such that

(M.1) $M_p^2 \leqslant M_{p-1} M_{p+1}$ for $p \in \mathfrak{N}$;

(M.2)′ $M_{p+1} \leqslant AH^p M_p$ for some $A>0$ and $H>0$, $p \in \mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$;

(M.3)′ $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$.

Let $C(K)$ be the space of continuous functions f on K supplied with the norm

$$\|f\|_{C(K)} = \sup_{x \in K} \{|f(x)|\}.$$

Let $h>0$. Denote by X_h the space of infinitely differentiable (smooth) functions ϕ with $\operatorname{supp} \phi \subset K$ such that

$$\|\phi\|_{X_h} = \sup_{\alpha \in \mathfrak{N}_0^n} \left\{ \frac{h^{|\alpha|} \|D^\alpha \phi\|_{C(K)}}{M_{|\alpha|}} \right\} < \infty$$

and

$$\frac{h^{|\alpha|} \|D^\alpha \phi\|_{C(K)}}{M_{|\alpha|}} \to 0 \qquad \text{as} \quad |\alpha| \to \infty, \tag{1}$$

where $\mathfrak{N}_0^n$ is the set of n-tuples of nonnegative integers $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})$. The space X_h is a Banach space.

Spaces $\mathscr{D}_K^{(M_p)}$ and $\mathscr{D}^{(M_p)}(\mathcal{O})$ are defined as follows:

$$\mathscr{D}_K^{(M_p)} = \varprojlim_{j \to \infty} X_j \qquad \text{(see [2, pp. 44, 77])}.$$

$$\mathscr{D}^{(M_p)}(\mathcal{O}) = \varinjlim_{K \Subset \mathcal{O}} \mathscr{D}_K^{(M_p)},$$

where $\varinjlim$ and $\varprojlim$ denote inductive and projective (topological) limits. $K \Subset \mathcal{O}$ means that K belongs to the family of regular compact subsets of $\mathcal{O}$ with the union $\mathcal{O}$. There are many "good" properties of these spaces [2]; for example, these spaces are separable and complete.

The strong dual of the space $\mathscr{D}^{(M_p)}(\mathcal{O})$, $(\mathscr{D}^{(M_p)}(\mathcal{O}))'$, is the space of Beurling ultradistributions.

Denote by Y the space of sequences $(\psi_\alpha; \alpha \in \mathfrak{N}_0^n)$, from $C(K)$ for which

$$\|(\psi_\alpha)\|_Y = \sup_{\alpha \in \mathfrak{N}_0^n} \{\|\psi_\alpha\|_{C(K)}\} < \infty \qquad \text{and} \qquad \lim_{|\alpha| \to \infty} \|\psi_\alpha\|_{C(K)} = 0.$$

(We enumerate the set $\mathfrak{N}_0^n$ in the usual way. For example, $\mathfrak{N}_0^2$ is enumerated as follows: $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots$). The space Y is a separable Banach space.

Let d_h, $h > 0$, be the mapping from X_h into Y defined by

$$\phi \mapsto d_h(\phi) = \left(\frac{h^{|\alpha|}(-1)^{|\alpha|} D^\alpha \phi}{M_{|\alpha|}} ; \alpha \in \mathfrak{N}_0^n \right).$$

Denote by $\tilde{X}_h$ the set $d_h(X_h) \subset Y$.

LEMMA 1. *We have*

$$\mathscr{D}_K^{(M_p)} \xrightarrow{i_1} X_h \xrightarrow{d_h} \tilde{X}_h \xrightarrow{i_2} Y,$$

where i_1 and i_2 are continuous inclusion mappings and d_h is an isometry.

Proof. Let $\phi \in \mathscr{D}_K^{(M_p)}$ and $l > h$. We have

$$\frac{h^{|\alpha|} \|D^\alpha \phi\|_{C(K)}}{M_{|\alpha|}} \leqslant (h/l)^{|\alpha|} \|\phi\|_{X_l} \to 0 \qquad \text{as } |\alpha| \to \infty.$$

This implies that i_1 is continuous. Other assertions trivially follow.

3. RANDOM LINEAR FUNCTIONALS ON $\mathscr{D}^{(M_p)}(\mathcal{O})$

In the following theorem we shall use the ideas of the proof of [2, Theorem 8.1] and [1, Theorem 6.1].

THEOREM 2. *Let f be an rlf on $\Omega \times \mathscr{D}^{(M_p)}(\mathcal{O})$. For every regular, relatively compact open subset G of $\mathcal{O}$ and every $\varepsilon > 0$ there exist $B \in \mathfrak{A}$ and finite random Radon measures $v_\alpha(\omega, dx)$ on $C(G)$, $\alpha \in \mathfrak{N}_0^n$, such that:*

(i) $\mu(B) \geqslant 1 - \varepsilon$.

(ii) *For every $\omega \in B$ there are $L > 0$ and $c > 0$ such that*

$$\|v_\alpha(\omega, dx)\|_{C'(G)} < cL^{|\alpha|}/M_{|\alpha|}, \quad \alpha \in \mathfrak{N}_0^n.$$

(iii) *For every $\omega \in B$ and $\phi \in \mathscr{D}^{(M_p)}(G)$*

$$f(\omega, \phi) = \sum_{\alpha \in \mathfrak{N}_0^n} \langle D^\alpha v_\alpha(\omega, dx), \phi(x) \rangle. \tag{2}$$

Moreover, for every $\omega \in B$ the series in (2) converges to $f(\omega, \cdot)$ in the sense of strong topology in $(\mathscr{D}^{(M_p)}(G))'$.

Note that $\| \ \|_{C'(G)}$ denotes the dual norm in $C'(G)$ and that in (2), D^α is the distributional derivative. The pairing between an element from $(\mathscr{D}^{(M_p)}(G))'$ and an element from $\mathscr{D}^{(M_p)}(G)$ is denoted by $\langle \cdot, \cdot \rangle$.

Proof. Since G is regular, relatively compact open subset of $\mathcal{O}$, the closure, $\bar{G}$, is regular compact subset of $\mathcal{O}$. Denote $\bar{G}$ by K. Let f_1 be restriction of the function f to the set $\Omega \times \mathscr{D}_K^{(M_p)}$. f_1 is an rlf on $\Omega \times \mathscr{D}_K^{(M_p)}$.

We shall show that f_1 can be represented in the form (2) in the strong topology of $(\mathscr{D}_K^{(M_p)})'$. Since the inclusion mapping $\mathscr{D}^{(M_p)}(G) \mapsto \mathscr{D}_K^{(M_p)}$ is the continuous one and so maps bounded sets into bounded sets, it follows that (2) holds in the strong topology of $(\mathscr{D}^{(M_p)}(G))'$.

From the definition of the topological structure of $\mathscr{D}_K^{(M_p)}$ it follows that for any $\omega \in \Omega$ there are $c_\omega > 0$ and $j(\omega) \in \mathfrak{N}$ such that

$$|f_1(\omega, \phi)| < c_\omega \|\phi\|_{X_{j(\omega)}}, \qquad \phi \in \mathscr{D}_K^{(M_p)}. \tag{3}$$

As in [1, Theorem 6.1] we put

$$A_N(\phi) = \{\omega \in \Omega;\ |f_1(\omega, \phi)| < N\|\phi\|_{X_N}\}, \qquad \phi \in \mathscr{D}_K^{(M_p)},\ N \in \mathfrak{N}$$

and

$$A_N = \bigcap_{\phi \in \mathscr{D}_K^{(M_p)}} A_N(\phi), \qquad N \in \mathfrak{N}.$$

Since $\mathscr{D}_K^{(M_p)}$ is separable, we have

$$A_N = \bigcap_{\phi \in \Lambda} A_N(\phi) \in \mathfrak{A},$$

where Λ is a dense denumerable subspace of $\mathscr{D}_K^{(M_p)}$. Thus, from

$$\Omega = \bigcup_{N=1}^{\infty} A_N \qquad \text{and} \qquad A_N \subset A_{N+1}, \qquad N \in \mathfrak{N},$$

we obtain that for a given $\varepsilon > 0$ there exists $r \in \mathfrak{N}$ such that $\mu(A_r) \geq 1 - \varepsilon$. If we put $B = A_r$, we obtain that for any $\omega \in B$ and $\phi \in \mathscr{D}_K^{(M_p)}$,

$$|f_1(\omega, \phi)| < r\|\phi\|_{X_r}.$$

Define on $\mathscr{D}_K^{(M_p)}$,

$$\phi \mapsto \tilde{f}_1(\omega, \phi) = \begin{cases} f_1(\omega, \phi), & \omega \in B \\ 0, & \omega \notin B. \end{cases}$$

We put

$$S(\omega) = \sup_{\substack{\phi \in \mathscr{D}_K^{(M_p)} \\ \|\phi\|_{X_r} \leqslant 1}} \{|\tilde{f}_1(\omega, \phi)|\}.$$

We have

$$S(\omega) = \sup_{\substack{\phi \in \Lambda \\ \|\phi\|_{X_r} \leqslant 1}} \{|\tilde{f}_1(\omega, \phi)|\}.$$

So, $S(\omega)$ is a measurable function and $S(\omega) \leqslant r$.

$\mathscr{D}_K^{(M_p)}$ is a subspace of X_r. By the probabilistic Hahn–Banach theorem [3, Theorem 2, p. 1154] $\tilde{f}_1$ can be extended on X_r to be an rlf on $\Omega \times X_r$. We denote this extension by f_2. Then

$$|f_2(\omega, \phi)| \leqslant S(\omega)\|\phi\|_{X_r}, \qquad \phi \in X_r,\ \omega \in \Omega \tag{4}$$

holds.

Since the mapping d_r is an isometry of X_r onto $\tilde{X}_r$ (see Section 2), the mapping from $\Omega \times \tilde{X}_r$ into the set of complex numbers C defined by

$$(\omega, (\tilde{\phi}_\alpha)) \mapsto f_2(\omega, \phi), \qquad \text{where } \phi = d_r^{-1}((\tilde{\phi}_\alpha)),$$

is an rlf on $\Omega \times \tilde{X}_r$. Denote this rlf by F. We have

$$F(\omega, (\tilde{\phi}_\alpha)) = f_2(\omega, \phi), \qquad \omega \in \Omega,\ \phi \in X_r,\ (\tilde{\phi}_\alpha) = d_r(\phi).$$

By the probabilistic Hahn–Banach theorem F can be extended on $\Omega \times Y$. Denote this extension by $\tilde{F}$. $\tilde{F}$ is an rlf such that

$$|\tilde{F}(\omega, (\psi_\alpha))| \leqslant S(\omega)\|(\psi_\alpha)\|_Y \qquad \text{for every } \omega \in \Omega \text{ and every } (\psi_\alpha) \in Y.$$

For every $\omega \in \Omega$, $\tilde{F}(\omega, \cdot)$ is a continuous linear functional on Y and therefore $\tilde{F}(\omega, \cdot)$ is of the form

$$\tilde{F}(\omega, \cdot) = \sum_{\alpha \in \mathfrak{N}_0^n} F_\alpha(\omega, \cdot),$$

and

$$\|\tilde{F}(\omega, \cdot)\|_{Y'} = \sum_{\alpha \in \mathfrak{N}_0^n} \|F_\alpha(\omega, \cdot)\|_{C'(K)} = S(\omega),$$

where $F_\alpha(\omega, \cdot)$, $\alpha \in \mathfrak{N}_0^n$, are continuous linear functionals on subspaces $Y_\alpha \subset Y$, $\alpha \in \mathfrak{N}_0^n$, where

$$Y_\alpha = \{(\psi_\beta; \beta \in \mathfrak{N}_0^n),\ \psi_\beta \in C(K),\ \psi_\beta \equiv 0 \text{ for } \beta \in \mathfrak{N}_0^n \backslash \{\alpha\}\}.$$

For every $\alpha \in \mathfrak{N}_0^n$ the space Y_α is isometric to $C(K)$. We shall denote by $[\phi]_\alpha$ the element from Y_α that corresponds to $\phi \in C(K)$. Since $F(\omega, \cdot)|_{Y_\alpha} = F_\alpha(\omega, \cdot)$, $\omega \in \Omega$, it follows that F_α are rlf on $\Omega \times Y_\alpha$, $\alpha \in \mathfrak{N}_0^n$, i.e., on $\Omega \times C(K)$.

Now [1, Lemma 5.2] directly implies that for every $\alpha \in \mathfrak{N}_0^n$ there exists one and only one finite random Radon measure $\nu_\alpha(\omega, dx)$ such that

$$F_\alpha(\omega, [\psi]_\alpha) = \int_K \psi(x)\, \nu_\alpha(\omega, dx), \qquad \omega \in \Omega,\ \psi \in C(K).$$

Thus, for $\omega \in B$ and $\phi \in \mathscr{D}_K^{(M_p)}$,

$$\begin{aligned} f_1(\omega, \phi) &= \tilde{f}_1(\omega, \phi) = f_2(\omega, \phi) \\ &= \tilde{F}\left(\omega, \left(\frac{r^{|\alpha|}(-1)^\alpha D^\alpha \phi}{M_{|\alpha|}}\right)\right) \\ &= \sum_{\alpha \in \mathfrak{N}_0^n} \int_K \frac{r^{|\alpha|}(-1)^\alpha}{M_{|\alpha|}} D^\alpha \phi(x)\, \nu_\alpha(\omega, dx) \\ &= \sum_{\alpha \in \mathfrak{N}_0^n} \left\langle \frac{r^{|\alpha|}}{M_{|\alpha|}} D^\alpha \nu_\alpha(\omega, dx), \phi(x) \right\rangle \\ &= \left\langle \sum_{\alpha \in \mathfrak{N}_0^n} D^\alpha \tilde{\nu}_\alpha(\omega, dx), \phi(x) \right\rangle, \end{aligned}$$

where

$$\tilde{\nu}_\alpha(\omega, dx) = \frac{r^{|\alpha|}}{M_{|\alpha|}} \nu_\alpha(\omega, dx),$$

and

$$\sum_{\alpha \in \mathfrak{N}_0^n} \left\| \frac{\tilde{\nu}_\alpha(\omega, dx)\, M_{|\alpha|}}{r^{|\alpha|}} \right\|_{C'(K)} < \infty.$$

This implies (ii) and completes the proof.

4. Random Linear Functionals on $\Omega \times \operatorname{Exp} \mathscr{A}$

The definition of the space $\operatorname{Exp} \mathscr{A}$ is given in [4]. Let I be an open interval in $\mathfrak{R}$, $L^2(I)$ a space of square integrable functions on I, $\mathscr{R}$ a linear differential self-adjoint operator of the form

$$\mathscr{R} = \theta_0 D^{n_1} \theta_1 \cdots D^{n_\nu} \theta_\nu,$$

where η_k, $k=1, ..., v$ are nonnegative integers, θ_k, $k=0, ..., v$ are smooth functions without zeros on I. Suppose that there exist a sequence of real numbers $(\lambda_n; n \in \mathfrak{N}_0)$ and a sequence of smooth functions $(\psi_n; n \in \mathfrak{N}_0)$ such that $\mathscr{R}\psi_n = \lambda_n \psi_n$, $n \in \mathfrak{N}_0$. Furthermore, suppose that $(|\lambda_n|; n \in \mathfrak{N}_0)$ is a non-decreasing sequence which tends to infinity and that $(\psi_n; n \in \mathfrak{N}_0)$ forms an orthonormal base of $L^2(I)$. We put

$$\tilde{\lambda}_n = \begin{cases} |\lambda_n| & \text{if } \lambda_n \neq 0,\ n \in \mathfrak{N}_0, \\ 1 & \text{if } \lambda_n = 0,\ n \in \mathfrak{N}_0, \end{cases}$$

and

$$\exp_p \tilde{\lambda}_n = \underbrace{\exp(\exp \cdots (\exp}_{p} \tilde{\lambda}_n) \cdots).$$

For a given $p \in \mathfrak{N}$ the space $\operatorname{Exp} \mathscr{A}_{p,p}$ is defined in [4] as the space of all $\phi \in L^2(\mathfrak{R}^n)$, $\phi =^2 \sum_{n=0}^{\infty} a_n \psi_n$, such that

$$\gamma_p(\phi) = \sum_{n=0}^{\infty} |a_n|^2 (\exp_p \tilde{\lambda}_n)^{2p} < \infty.$$

(Here $=^2$ means the equality in the sense of L^2-norm.) The sequence $(\operatorname{Exp} \mathscr{A}_{p,p};\ p \in \mathfrak{N})$ is a projective reduced compact sequence of Banach spaces (see [2, p. 33]), so that

$$\operatorname{Exp} \mathscr{A} = \varprojlim_{p \to \infty} \operatorname{Exp} \mathscr{A}_{p,p}$$

is complete. The set of polynomials of the form $\sum_{k=0}^{n} (a_k + ib_k) \psi_k$, $n \in \mathfrak{N}_0$, where a_k and b_k are rational numbers, is a dense subspace of $\operatorname{Exp} \mathscr{A}$ and of $\operatorname{Exp} \mathscr{A}_{p,p}$, $p \in \mathfrak{N}$. This means that $\operatorname{Exp} \mathscr{A}$ and $\operatorname{Exp} \mathscr{A}_{p,p}$, $p \in \mathfrak{N}$ are separable. We note that $\operatorname{Exp} \mathscr{A}$ is a dense subspace of the spaces $\exp_p \mathscr{A}$ which are studied in [5].

If f is an rlf on $\Omega \times \operatorname{Exp} \mathscr{A}$, then for any fixed $\omega \in \Omega$, $f(\omega, \cdot)$ is an element of $(\operatorname{Exp} \mathscr{A})'$. Thus, it is of the form

$$f(\omega, \cdot) = \sum_{n=0}^{\infty} b_n(\omega) \psi_n,$$

i.e.,

$$f(\omega, \phi) = \sum_{n=0}^{\infty} b_n(\omega) \langle \psi_n, \phi \rangle, \qquad \phi \in \operatorname{Exp} \mathscr{A}$$

$(\langle\psi_n, \phi\rangle = \int_{\mathfrak{R}} \psi_n(t)\, \phi(t)\, dt)$. Here $b_n(\omega) = f(\omega, \bar{\psi}_n)$, $n \in \mathfrak{N}$, and for some $r = r_\omega$

$$\sum_{n=0}^{\infty} |b_n(\omega)|\, (\exp_r \tilde{\lambda}_n)^{-2r} < \infty \qquad \text{(see [4, 5])}$$

($\bar{\psi}_n$ is the conjugate function for ψ_n).

From the definition of an rlf it follows that the functions $\omega \mapsto b_n(\omega)$, $n \in \mathfrak{N}_0$ are measurable. Now, by using the fact that

$$(\operatorname{Exp} \mathscr{A})' = \varinjlim_{p \to \infty} (\operatorname{Exp} \mathscr{A}_{p,p})' \qquad \text{(see [2, p. 33])},$$

in the same way as in Theorem 2 one can prove:

THEOREM 3. *If f is an rlf on $\Omega \times \operatorname{Exp} \mathscr{A}$, then for every $\varepsilon > 0$ there exists $B \in \mathfrak{A}$ with $\mu(B) \geqslant 1 - \varepsilon$ and there exist $k = k_\varepsilon$ such that for every $\omega \in B$ and every $\phi \in \operatorname{Exp} \mathscr{A}$*

$$f(\omega, \phi) = \sum_{n=0}^{\infty} b_n(\omega) \langle \psi_n, \phi \rangle, \tag{5}$$

where

$$b_n(\omega) = f(\omega, \bar{\psi}_n), \qquad n \in \mathfrak{N}_0,\ \omega \in B$$

and

$$\sum_{n=0}^{\infty} |b_n(\omega)|^2\, (\exp_k \tilde{\lambda}_n)^{-2k} < \infty.$$

Moreover, for every $\omega \in B$ the series in (5) converges to $f(\omega, \cdot)$ in the sense of strong topology in $(\operatorname{Exp} \mathscr{A})'$.

5. APPLICATION

Let $f(\cdot,\cdot) = \langle \sum_{n=0}^{\infty} b_n(\cdot)\, \psi_n, \cdot \rangle$ be an rlf on $\Omega \times \operatorname{Exp} \mathscr{A}$, and P be a polynomial such that the following condition holds:

If for some $\omega \in \Omega$ and $n \in \mathfrak{N}_0$ $b_n(\omega) \neq 0$, then $P(\lambda_n) \neq 0$.

Then the stochastic differential equation

$$P(\mathscr{R})\, u(\omega, \phi) = f(\omega, \phi), \qquad \omega \in \Omega,\ \phi \in \operatorname{Exp} \mathscr{A}, \tag{6}$$

where $P(\mathscr{R})\, u(\omega, \phi) = u(\omega, P(\mathscr{R})\, \phi)$, $\omega \in \Omega$, $\phi \in \operatorname{Exp} \mathscr{A}$, has the solution

$$(\omega, \phi) \mapsto u(\omega, \phi) = \left\langle \sum_{n=0}^{\infty} \frac{b_n(\omega)}{P(\lambda_n)} \psi_n, \phi \right\rangle$$

(if for some ω, $b_n(\omega) = 0$ and $P(\lambda_n) = 0$ we put $b_n(\omega)/P(\lambda_n) = 0$). The solution u is an rlf for which Theorem 3 implies

For every $\varepsilon > 0$ there exists $B \in \mathfrak{A}$ and $k = k_\varepsilon$ such that $\mu(B) \geqslant 1 - \varepsilon$ and $\sum |u_n(\omega)|^2 (\exp_k \tilde{\lambda}_n) - 2k < \infty$, where $u_n(\omega) = b_n/P(\lambda_n)$, $n \in \mathfrak{N}_0$, $\omega \in B$. (*)

If we denote by E_1^k the differential operator of infinite order

$$E_1^k = (e^{\mathscr{R}})^k = e^{k\mathscr{R}} = \sum_{m=0}^{\infty} \frac{k^m \mathscr{R}^m}{m!}$$

and by

$$E_p^k = \exp k \underbrace{(\exp \cdots \exp \mathscr{R})}_{p}$$

$$= \sum_{m_1=0}^{\infty} \frac{k^{m_1}}{m_1!} \sum_{m_2=0}^{\infty} \frac{m_1^{m2}}{m_2!} \cdots \sum_{m_p=0}^{\infty} \frac{m_{p-1}^{m_p}}{m_p!} \mathscr{R}^{m_p},$$

then by the same arguments we can discuss and solve the stochastic differential equation of infinite order

$$P_0(\mathscr{R})\, u + P_1(E_1)\, u + \cdots + P_s(E_s)\, u = Hu = f.$$

More precisely this equation is of the form

$$Hu(\omega, \phi) \equiv u(\omega, H\phi) = f(\omega, \phi), \qquad \omega \in \Omega,\ \phi \in \operatorname{Exp} \mathscr{A},$$

where f is a given rlf from $(\operatorname{Exp} \mathscr{A})'$ and $P_0, \ldots, P_s$ are arbitrary polynomials (see [4]).

Now we shall consider the stochastic differential equation

$$\begin{aligned} &\frac{\partial u(\omega, t, \phi)}{\partial t} = P(\mathscr{R})\, u(\omega, t, \phi), \qquad \omega \in \Omega,\ t \in [0, \infty),\ \phi \in \operatorname{Exp} \mathscr{A}, \\ &u(\omega, \phi) = f(\omega, \phi), \qquad \omega \in \Omega,\ \phi \in \operatorname{Exp} \mathscr{A}, \end{aligned} \tag{7}$$

where f is given rlf on $\Omega \times \operatorname{Exp} \mathscr{A}$.

Assume that $u(\omega, \cdot, \phi)$ is a smooth function on $[0, \infty)$ for every $\omega \in \Omega$, $\phi \in \operatorname{Exp} \mathscr{A}$, and that for any $\omega \in \Omega$, $t \in [0, \infty)$ and $\phi \in \operatorname{Exp} \mathscr{A}$:

$$P(\mathscr{R})\, u(\omega, t, \phi) = u(\omega, t, P(\mathscr{R})\, \phi).$$

By using the results from [4] we obtain that

$$(\omega, t, \phi) \mapsto u(\omega, t, \phi) = \left\langle \sum_{n=0}^{\infty} (\exp(P(\lambda_n)\, t))\, b_n(\omega)\, \psi_n, \phi \right\rangle,$$

$$b_n(\omega) = f(\omega, \bar{\psi}_n), \qquad n \in \mathfrak{N}_0,\ \omega \in \Omega,$$

is a solution of (7). For coefficients $u_n(\omega, t) = (\exp(P(\lambda_n)\, t)\, b_n(\omega)$, $n \in \mathfrak{N}_0$, condition $(*)$ holds if t belongs to a bounded set in $[0, \infty)$ (and B and k now depend on this bounded set).

In the same way we can discuss and solve the infinite order stochastic differential equation of the form

$$\frac{\partial u(\omega, t, \phi)}{\partial t} = (P_0(\mathscr{R}) + P_1(E_1) + \cdots + P_s(E_s))\, u(\omega, t, \phi),$$

$$\omega \in \Omega,\ t \in [0, \infty),\ \phi \in \operatorname{Exp} \mathscr{A}, \tag{7'}$$

$$u(\omega, 0, \phi) = f(\omega\, \phi), \qquad \omega \in \Omega,\ \phi \in \operatorname{Exp} \mathscr{A}$$

(with notations given above), where f is a given rlf on $\Omega \times \operatorname{Exp} \mathscr{A}$.

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